Section 4.8

Newton's Method

Newton's Method

Newton's Method is a procedure which uses successive tangent lines to find a numerical approximation to the equation $f(x) = 0$.

Numerical approximations are important as it is often impossible to find exact solutions.

Polynomials (Optional)

You're used to being able to solve equations because you haven't yet experienced the pain of finding the roots of higher order polynomials.

The quadratic equation is not too bad...

$$ax^2 + bx + c = 0 \qquad \Longrightarrow \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

but what about higher-degree polynomials?

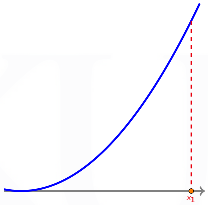
$$ax^3 + bx^2 + cx + d = 0 \qquad \Longrightarrow \qquad x = \ ?$$

$$ax^4 + bx^3 + cx^2 + dx + e = 0 \qquad \Longrightarrow \qquad x = \ ?$$

There exist (complicated!) formulas for the cubic and the quartic, but not for degrees 5 and higher (as proved by Abel in 1824).

Newton's Method

Newton's Method uses tangent lines to find better and better approximations of a value c which satisfies $f(c) = 0$.



(1) Guess a value of the root c. Call this guess x_0.

(2) Take the tangent line of f at x_0

$$y = f(x_0) + f'(x_0)(x - x_0)$$

(3) Unless horizontal, the tangent line will cross the x-axis at some point, $(x_1, 0)$.

$$0 = f(x_0) + f'(x_0)(x_1 - x_0) \qquad \Rightarrow \qquad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Repeat this process to find a new approximation of the root c:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Newton's Method

To approximate a root of $f(x)$, choose an initial value x_0 which is **not** a critical point. Generate successive approximations of the root through the equation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example 1: Approximate $\sqrt{5}$ using $f(x) = x^2 - 5$ and $x_0 = 2$. $f'(x) = 2x$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.25$$

$$x_2 = 2.25 - \frac{(2.25)^2 - 5}{2(2.25)} \approx 2.2361$$

Newton's Method quickly approximates roots:

$$\sqrt{5} \approx 2.23607...$$

Example 2 (Optional):

Approximate a root of $f(x) = x^5 - x - 1$ using Newton's Method.

Solution: Since $f(1) = -1$ and $f(2) = 29$, the Intermediate Value Theorem says that there is a root c in $(1, 2)$. Note that $f'(x) = 5x^4 - 1$.

$$x_0 = 1$$

$$x_1 = 1 - \frac{(1)^5 - (1) - 1}{5(1)^4 - 1} = 1.25$$

$$x_2 = 1.25 - \frac{f(1.25)}{f'(1.25)} \approx 1.1785$$

$$x_3 \approx 1.1675 \ , \ x_4 \approx 1.1673$$

$$x_0 = 2$$

$$x_1 = 2 - \frac{(2)^5 - (2) - 1}{5(2)^4 - 1} \approx 1.6329$$

$$x_2 \approx 1.3731 \ , \ x_3 \approx 1.2236$$

$$x_4 \approx 1.1727 \ , \ x_5 \approx 1.1674$$

Sometimes, Newton's Method does not succeed in approximating a root.

(I) If x_0 is a critical point, then Newton's Method fails at step 1.

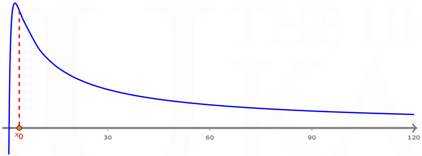
$$\text{If } f'(x_0) = 0 \text{ or does not exist, then } x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \text{ is not defined.}$$

(II) If the graph of f is very complicated, Newton's Method can take a long time to find a root, or can even bounce back and forth around the graph without ever converging.

(III) Sometimes Newton's Method approaches a horizontal asymptote.

Example 3 (Optional):

Let $f(x) = \dfrac{\ln(x)}{x}$ and take $x_0 = 4$. Note that $f'(x) = \dfrac{1 - \ln(x)}{x^2}$.



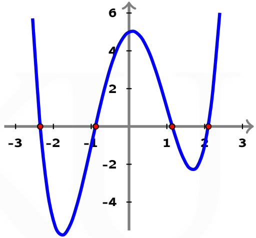
$x_1 \approx 18.355$

$x_2 \approx 46.32$

$x_3 \approx 108.98$

If your first choice for x_0 does not work, choose a different x_0! In this problem the starting point has to be **very** close to the root.

If f has multiple roots, then different choices of x_0 may lead to different roots.



Example 4 (Optional):

$f(x) = x^4 - 6x^2 + x + 5$

$f'(x) = 4x^3 - 12x + 1$

$x_0 = 1$

$x_1 \approx 1.1429$

$x_2 \approx 1.1446$

$x_3 \approx 1.1446$

Closing in on the root $c \approx 1.14465$

$x_0 = 0$

$x_1 = -5$

$x_2 \approx -3.918$

$x_3 \approx -3.167$

Closing in on the root $c \approx -2.34943$